On Construction of a Digital Plane using GCD Algorithm

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Abstract

A digital arc is the digitization of a straight line segment if and only if it has the "chord property". This result is used to derive several regularity properties of digitization of straight line segments. The idea of 2D digital straight line segment has been extended to 3D digital straight line segment which has further been utilized for the characterization of a digital plane. A digital plane can be represented as a net code. Each and every row or column of this net code has a particular definite pattern which gets shifted and repeated in the next row or column. This pattern is basically a 2D chain Code. In this paper, it has been shown that, only the norm of a plane is sufficient to derive the horizontal and vertical pattern and corresponding shifts. Moreover, both horizontal and vertical patterns and any one of the corresponding shifts are sufficient to generate a digital plane. This concept may be used in developing a deep learning model which may be trained to identify digital planes given a set of voxels on the surface of a 3D objects.

Preliminaries

A plane is uniquely determined by a point on the plane and a vector perpendicular to the plane. Such a vector is said to be normal to the plane. Given a point $P_0 = (x_0, y_0, z_0)$ and a normal $\vec{n} = \langle a, b, c \rangle$ to a plane, a point $P = (x, y, z)$ will be on the plane if $\vec{P_0P}$ is perpendicular to $\vec{n}$ that is

$$\vec{n} \,.\vec{P_0P} = 0 \tag{1}$$

This is known as vector equation of a plane. Switching to coordinates, we get

$$\langle a, b, c \rangle . \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This is usually written as

$$ax + by + cz + d = 0 \tag{2}$$

where $d = -ax_0 - by_0 - cz_0$. This is known as the scalar equation of a plane.

Here are two assumptions for the benefit of the analysis of the 3D plane's quadrant step code Without loss of generality, the following assumptions are made to simplify the analysis.

1. The plane passes through the origin the plane equation becomes simpler and reduces to

$$z = -\frac{a}{c}x - \frac{b}{c}y \tag{3}$$

2. The norm $\langle a, b, -c \rangle$ (a, b, and c are positive integers) is such that $c \geqslant a \geqslant b$, then Eqn. 3 becomes

$$z = \frac{a}{c}x + \frac{b}{c}y \tag{4}$$

which is the standard situation equation of plane [2], i.e., $0 \leqslant \frac{b}{c} \leqslant \frac{a}{c} \leqslant 1$.

It may be noted here that by suitable axes transformation and translation of the origin, any plane may be equivalently given by $z = px + qy + r, 0 \leqslant q \leqslant p \leqslant 1$, where $p = \frac{a}{c}$, $q = \frac{b}{c}$, and $r = \frac{d}{c}$. Clearly $D(P)$ contains $(n + 1)^2$ points which are represented by $(n + 1) \times (n + 1)$ matrix v where $v(i, j) = k$

Generation of Plane Voxels

For a norm $\langle a, b, -c \rangle$ such that $c \geqslant a \geqslant b$, the slope along yz-plane, Sl_{yz} is $\frac{b}{c}$ and the slope along xz-plane, Sl_{xz} is $\frac{a}{c}$. We can generate corresponding chain codes for these two slopes using the idea of continued fraction. Now, the 2DSS canonical pattern along yz-plane, P_{yz} and the 2DSS canonical pattern along xz-plane, P_{xz} will remain same for each and every row and column respectively, but will be shifted in each next row or column. If the plane equation is given by $z = \frac{a}{c} \times x + \frac{b}{c} \times y$, then the real value of the z coordinates are given by the Table 1.

Table 1: Real Z Coordinate Values of the Plane Quadrant

x \ y	0	1	2	3
0	0	$\frac{b}{c}$	$\frac{2\times b}{c}$	$\cdots$
1	$\frac{a}{c}$	$\frac{a}{c} + \frac{b}{c}$	$\frac{a}{c} + \frac{2\times b}{c}$	$\cdots$
2	$\frac{2\times a}{c}$	$\frac{2\times a}{c} + \frac{b}{c}$	$\frac{2\times a}{c} + \frac{2\times b}{c}$	$\cdots$
3	$\frac{3\times a}{c}$	$\frac{3\times a}{c} + \frac{b}{c}$	$\frac{3\times a}{c} + \frac{2\times b}{c}$	$\cdots$
4	$\vdots$	$\vdots$	$\vdots$	$\ddots$

These real values are converted to integers based on its proximity to the floor or ceiling value, which is exactly the same as the Bresenham's method of line drawing. Now, if the *Quadrant Step Code* [1] is generated, it is seen that the *row or column wise step code* is same in each row or column respectively, except that the step code gets shifted.

Shift

The slope $Sl_{yz} = \frac{b}{c}$, along yz-plane gives a binary pattern, P_{yz}, of length c, for a digital straight line. This pattern repeats itself to give the straight line of slope Sl_{yz} in yz-plane. Similarly, the binary pattern P_{xz} of length c corresponding to slope $Sl_{xz} = \frac{a}{c}$ indicates a straight line along xz-plane. The algorithm creates a finite digital plane quadrant of size $c \times c$ with origin at one of its corners. From equation 4 the real value of z is 0 which is an integer number, for $x = 0$ and $y = 0$. So the digitized value of z will be 0 only. Now, if we increase x and y within the range $0 \leqslant x, y \leqslant c$, the immediate next occurrence of such an integer z value is the point from where the pattern of the step code starts freshly. This is because of the fixed slope along yz-plane or xz-plane and the grid-intersection method of digitization. An integer z value may occur in the next line of the plane quadrant or in a line later on.

In each row, the pattern P_{yz} gets shifted with respect to the previous row. We consider it as a *right circular shift*. The total shift may be distributed in a single row or more than one row. Same property is applicable for the shift in the xz-plane too, i.e., in each column the pattern P_{xz} gets shifted with respect to the previous column. This is also considered as *right circular shift*. The total shift may be distributed in a single column or more than one column.

Theorem 1 *If the total shifts of the 2DSS canonical patterns along yz-plane and xz-plane are denoted by S_{yz} and S_{xz} respectively, then the two general equations stated below are true for S_{yz} and S_{xz}.*

$$(a \times gcd(b, c) + b \times S_{yz})\%c = 0 \tag{5}$$

$$(a \times S_{xz} + b \times gcd(a, c))\%c = 0 \tag{6}$$

Proof: Origin is the point where for the first time, the real value of z is an integer, without rounding off. As we construct the plane quadrant by increasing the coefficient of $\frac{a}{c}$, which is x and the coefficient of $\frac{b}{c}$, which is y, within the range $0 \leqslant (x, y) \leqslant c$, we try to find the immediate next such occurrence where the real z value of the equation 4 is an integer number, without rounding off. This is the point from where the canonical pattern of the step code repeats itself. To find out this point, for S_{yz}, the coefficient of $\frac{a}{c}$ should be such that the value of the equation 4 becomes closest to an integer number. For that it should be $gcd(b, c)$. Similarly, for shift S_{xz} the coefficient of $\frac{b}{c}$ should be $gcd(a, c)$.

Hence average shift S_{yz} in each line, SP_{yz} will be $S_{yz}/gcd(b, c)$ and average shift S_{xz} in each line, SP_{xz} will be $S_{xz}/gcd(a, c)$.

Corollary 1 *Average shift in the pattern P_{yz} will be $S_h/gcd(b, c)$ and average shift in the pattern P_{xz} will be $S_v/gcd(a, c)$.*

References

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