Circular Partial Words and Arrays

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Abstract

Words and arrays over a finite alphabet are natural objects in several research areas including coding theory, theory of algorithms, artificial intelligence, embedded system, automata and formal language theory. DNA molecules turn out to be genetic instructions transporter. The strands of DNA molecules are considered as finite strings and are utilized for encoding details in DNA computation. The process of determining the order of nucleotides in DNA is termed as DNA sequencing. The development of rapid DNA sequencing methods has significantly stimulated medical and biological discovery and research. Perception of DNA sequences has become an essential requisite for fundamental biological research, and in various applied fields such as virology, biotechnology, medical diagnosis, biological systematics and forensic biology. In DNA sequencing, some unseen or missing part of information may exist. This can be revealed by positions denoting do not care symbols in a word. Thus, instead of complete words, partial words are considered in the comparison of genes. Related to DNA, it is of interest to investigate the characterizations of partial words. In order to analyse the difference in gene expressions between wild type and signaling mutants, partial DNA arrays are also utilized. The stucture of DNA molecules are not only linear but also circular. Motivated by the wide applications of partial words and circular words in DNA sequencing, we study circular partial language containing set of circular partial words and obtain the automata characterization of circular partial language. Also we establish their closure properties and deal with an operation termed as length product. The role of artificial intellegence in circular DNA sequencing is discussed. The implementation used in this work is a generalization to deal with circular sequences with holes. Further we extend circular partial words to circular partial arrays and discuss their combinatorial properties such as conjugacy, primitivity and periodicity.

Introduction

The field of combinatorics on words [de Luca 1999] bridges mathematics and computer science by focusing on combinatorial properties. Both combinatorics on words and the study of formal languages are related to each other since both the fields explore various properties of words such as types of words, structures, languages, visual representation of words and languages etc. The study of combinatorial properties of words can be traced back from the birth of twentieth century by the work of Axel Thue [Trygve Nagell and Thalberg 1977]. M. Lothaire [Lothaire 1997] in 1983 recognized Thue's work and introduced the first book on combinatorics on words which turned to be the stimulus for recent works on partial words [Machacek 2019, Nayak and Kapoor 2015].Partial words [Fischer and Paterson 1974] play a vital role in molecular biology since they are utilized in gene comparisons. In the context of gene comparison, the study of partial words was initiated by Berstel and Boasson [Berstel and Boasson 1999] and later extended by Blanchet Sadri [Blanchet-Sadri 2003, 2005, 2008, Blanchet-Sadri and Luhmann 2002].DNA molecules are nothing but genetic information carriers. In DNA computation, DNA strands are considered as finite strings (or words) and are utilized for encoding information. While encoding, some part of information may be unseen or missing which are revealed by using partial words that denotes the positions of the missing symbols in a word. Similar to words, the study of combinatorics on arrays (or two dimensional words or pictures) [Carpi and de Luca 1999, 2004, Giammarresi and Restivo 1997, S. Vijayachitra and Thomas 2017] is of fundamental interest for data processing and communication of DNA sequence assembly in molecular biology. An array is a two-dimensional rectangular arrangement of letters. In order to analyse the difference in gene expressions between wild type and signaling mutants, partial DNA arrays are also utilized. The structure of DNA molecules are not only linear but also circular (cyclic). Prokaryotic genomes are circular and many bacteria possess extra circular DNA molecules. [Fernandes, Pereira, and Freitas 2009]. Circular DNA molecules are present also in eukaryotic cells such as the chloroplast DNA in plant cells and the mitochondrial DNA (mtDNA) in both animal and plant cells. Thus every cell has some kind of genome that is circular which can be considered as a motivation for analyzing the properties of circular words. A circular word is a circular representation from a linear word by connecting its initial symbol after the final symbol. In other words, the set of all rotations (or conjugates) of a linear word is termed as circular word. The study of circular words avoiding patterns was introduced by Currie and Fitzpatrick [Currie and Fitzpatrick 2002]. Hegedüs and Nagy [Hegedüs and Nagy 2013, 2014] defined two representations of circular words namely iterative representation with tuples and graphical representation with trees such that the former is an encoding that utilizes the flexible properties of circular words and the latter represents the structure of different conjugates of a word and their relation to each other. Applications of circular words was discussed by Benoit Rittaud and Laurent Vivier [Rittaud and Vivier 2011]. In [R. Krishna Kumari and Dare 2020] circular words was extended to circular partial words and their properties were studied.

The work in [Blanchet-Sadri 2008] stresses major topics underlying the combinatorics of the emerging class of partial words such as conjugacy, periodicity, primitivity, coding, correlations, unavoidable sets etc. A hierarchy of circular languages was introduced in [Carton 1997] and a computation method called approximation by chains for decomposing a circular language into a strongly circular language was given. In [Dasharath Singh 2015], an operation on regular languages termed as length product was introduced. The study on partial words has been extended to partial arrays and primitivity of partial arrays has been studied [S. Vijayachitra and Thomas 2017]. circular partial words was introduced and certain properties were discussed in [R. Krishna Kumari and Dare 2020]. In this paper, we define circular partial automata to characterize circular partial languages and discusses its closure properties. Further we extend the notion of finite circular partial words to two-dimensional words defined as circular partial arrays and obtain the combinatorial properties such as conjugacy, primitivity and periodicity for circular partial arrays. The paper is organized as follows: In section the fundamental definitions pertaining to partial words and circular words are recalled. The closure properties of circular partial languages are discussed in section **??**. In section circular partial arrays are defined and its combinatorial properties are discussed.

Preliminaries

In this section, we briefly recall the standard definitions and notations regarding partial, circular words and languages and also circular partial words and languages. Let $\mathbb{N} := \{1, 2, 3, \ldots\}$ be the set of natural numbers. $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let the set Σ termed as *alphabet* represent a non-empty finite set of symbols (or letters). A *total word* or *string* is a sequence of letters over Σ. Σ^* denotes the set of all total words from Σ. $\Sigma^+ = \Sigma^* \setminus \{\lambda\}$. A language L is a subset of Σ^*. Let $w \in \Sigma^*$ by $|w|$ we denote the length of the word w. A *circular word* $(c)_\circ$ is a set of all conjugates derived from a total word c over Σ by linking its initial letter after the final letter. A *partial word* is a word that contains a number of "holes" or "wild card letters" (denoted as $\Diamond$) present anywhere in the sequence of letters of the word. The symbol $\Diamond$ does not belong to the alphabet Σ, but survives as a standby symbol for the unknown letter, for instance, $u = aa\Diamond b$ is a partial word with $|u| = 4$. A empty word λ is a partial word with holes such that the holes are compatible only with λ. Formally, a partial word u of length n over Σ is a partial function $u : \{0, 1, 2 \cdots, n-1\} \to \Sigma$. For $0 \le i < n$, if $u(i)$ is defined, then we say $i \in D(u)$ (the domain of u), otherwise $i \in H(u)$ (the set of holes). The following definition is used in order to represent the positions of the holes of the partial words. The companion of u, denoted by $u_\Diamond$ is the total function $u_\Diamond : \{0, 1, 2 \cdots, n-1\} \to \Sigma_\Diamond = \Sigma \cup \{\Diamond\}$ defined by

$$u_\Diamond(i) = \begin{cases} u(i) & \text{if } i \in \text{D(u)} \\ \diamond & \text{if } i \in \text{H(u)}. \end{cases}$$

The *length product of partial words* is defined as $u_1 \otimes u_2 = u_1 \cdot u_2$ if and only if $|u_1| = |u_2|$ and (u_1, u_2) is an ordered pair, where u_1 and u_2 are partial words over $\Sigma_\Diamond = \Sigma \cup \{\Diamond\}$. The *length product of partial languages* is defined by $L_1 \otimes L_2 = \{x \otimes y \mid x \in L_1, y \in L_2\}$ where $L_1, L_2 \subseteq \Sigma_\Diamond^*$. The partial word s is contained in the partial word t (denoted as $s \subset t$) if $D(s) \subset D(t)$ and $s(j) = t(j)\ \forall\ j \in D(s)$ and s and t are compatible (denoted by $s \uparrow t$) if there exists a partial word r such that $s \subset r$ and $t \subset r$. A partial word p is said to be $primitive$ if there exists no word q such that $p = q^m$ with $m \ge 2$. A partial word t over $\Sigma_\Diamond$ is $t - periodic$ if a non-negative integer t exists such that $i \equiv jmodt$ whenever $t(i) = t(j)\ \forall\ i, j \in$ D(t). A partial array A of size (m, n) over Σ is a partial function $A : Z_+^2 \to \Sigma$ where Z_+ is the set of all positive integers. For $1 \le i < m, 1 \le j < n$, if $A(i, j)$ is defined, then we say $(i, j) \in D(A)$ (the domain of A), otherwise $(i, j) \in H(A)$ (the set of holes). The following definition is used in order to represent the positions of the holes of the partial arrays. The companion of A, denoted by $A_\Diamond$ is the total function $A_\Diamond : z_+^2 \to \Sigma_\Diamond = \Sigma \cup \{\Diamond\}$ defined by

$$A_\Diamond(i, j) = \begin{cases} A(i, j) & \text{if } (i, j) \in \text{D(A)} \\ \diamond & \text{if } (i, j) \in \text{H(A)}. \end{cases}$$

For example, the partial array $A= \begin{bmatrix} a & \Diamond & b \\ b & a & a \\ a & b & \Diamond \end{bmatrix}$ is the companion of partial array A of size $(3, 3)$ where $D(A) = \{(1, 1), (1, 3).(2, 1), (2, 2), (2, 3), (3, 1), (3, 2)\}$ and $H(A) = \{(1, 2), (3, 3)\}$. The bijectivity of the map $A \to A_\Diamond$ allows to define the catenation of two partial arrays. Let $C_1 := [a_{i,j}]_{m,n}$ and $C_2 := [b_{i,j}]_{m',n'}$ be two non-empty partial arrays over Σ. By column catenation, we mean $C_1 \oplus C_2 =$

$$\begin{matrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} & b_{1,1} & b_{1,2} & \dots & b_{1,n'} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} & b_{2,1} & b_{2,2} & \dots & b_{2,n'} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} & b_{m',1} & b_{m',2} & \dots & b_{m',n'} \end{matrix}$$

where $m = m'$ and by row catenation, we mean $C_1 \ominus C_2 =$

$$\begin{matrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \\ b_{1,1} & b_{1,2} & \dots & b_{1,n'} \\ b_{2,1} & b_{2,2} & \dots & b_{2,n'} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m',1} & b_{m',2} & \dots & b_{m',n'} \end{matrix}$$

where $n = n'$.

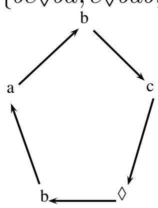
Circular Partial Languages

Here we define circular partial language and discuss the closure properties. Throughout the section we consider partial words consisting a single hole.

Definition 0.1 *The finite circular partial word over $\Sigma_\lozenge$ denoted as $(c_\lozenge)_\circ$ is the set of all possible conjugates of $c_\lozenge$ where $c_\lozenge$ represents any finite linear partial word. $|(c_\lozenge)_\circ|$ denotes the length of $(c_\lozenge)_\circ$ which is equal to the total count of conjugates of $c_\lozenge$.*

Example 0.2 *Let $c_\lozenge = bc\lozenge ba$ be a partial word over $\Sigma_\lozenge$. Then $(c_\lozenge)_\circ$ with $|(c_\lozenge)_\circ| = 5$ is*

$$(bc\lozenge ba)_\circ = \{bc\lozenge ba, c\lozenge bab, \lozenge babc, babc\lozenge, abc\lozenge b\}$$



$Figure\ 1$

Definition 0.3 *Let $x, y \in \Sigma_\lozenge^*$. A language L over $\Sigma_\lozenge$ is a* circular partial language *iff*

- $\forall n \in \mathbb{N}_0,\ x \in L \Leftrightarrow x^n \in L$
- $xy \in L \Leftrightarrow yx \in L$

Example 0.4 *$\Sigma_\lozenge^* b \Sigma_\lozenge^*$ over $\Sigma_\lozenge = \{a, b\} \cup \{\lozenge\}$ is a circular partial language, that can also be expressed as $\Sigma_\lozenge^* \setminus \{a^*\}$.*

All circular partial languages are not regular. Here we consider only regular circular partial languages and study certain operations.

Definition 0.5 *A* circular partial finite automaton *in short CPA is a quintuple $M = (Q_\lozenge, \Sigma_\lozenge, I_\lozenge, \delta, F_\lozenge)$ where*

1. *$Q_\lozenge = Q \cup Q_h$ where Q is a finite states set, Q_h is a finite hole states set and $Q \cap Q_h = \varnothing$,*
2. *$I_\lozenge = Q_\lozenge$ is the initial states set,*
3. *$\Sigma_\lozenge$ is the input alphabet,*
4. *$\delta : Q_\lozenge \times \Sigma_\lozenge \to Q_\lozenge$ is the transition function defined as follows:*
   For all $x \in \Sigma$,
   $\delta(p, x) = q$ for any $p \in Q$, $q \in Q$,
   $\delta(p, \lozenge) = q$ for any $p \in Q_\lozenge$, $q \in Q_h$,
   $\delta(p, x) = q$ for any $p \in Q_h$, $q \in Q$ and
5. *$F_\lozenge = Q_\lozenge$ is the set of final states. Each unique final state in the set $F_\lozenge$ also represents each unique initial state in the set $I_\lozenge$.*

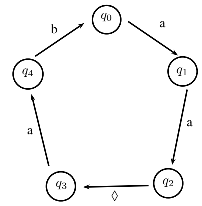
Example 0.6 *Consider the CPA $M = (Q_\lozenge, \Sigma_\lozenge, I_\lozenge, \delta, F_\lozenge)$ where*

1. *$Q_\lozenge = \{q_0, q_1, q_2, q_3, q_4\}$ such that the states $Q = \{q_0, q_1, q_2, q_4\}$ and $Q_h = \{q_3\}$,*
2. *$I_\lozenge = \{q_0, q_1, q_2, q_3, q_4\}$,*
3. *$\Sigma_\lozenge = \{a, b\} \cup \{\lozenge\}$,*
4. *$F_\lozenge = \{q_0, q_1, q_2, q_3, q_4\}$ and*
5. *the transition table defined as follows:*

$Table\ 1$

State / Symbol	a	b	$\lozenge$
q_0	q_1	-	-
q_1	q_2	-	-
q_2	-	-	q_3
q_3	q_4	-	-
q_4	-	q_0	-

The transition diagram for M is the following:



$Figure\ 2$

Then the circular partial finite automaton M recognizes the circular partial language $L(M) = \{(aa\lozenge ab)_\circ^n | n \geq 1\}$

Theorem 0.7 *If L_1 and L_2 over $\Sigma_\lozenge$ are regular circular partial languages, then*
(i) $L_1 \cap L_2$ is a regular circular partial language,
(ii) $L_1 \cup L_2$ is a regular circular partial language..

Proof: (i) Let L_1 and L_2 be any regular circular partial languages recognized by circular partial finite automata $M_1 = (R_\lozenge, I_1, \Sigma_\lozenge, \delta_1, F_1)$ and $M_2 = (S_\lozenge, I_2, \Sigma_\lozenge, \delta_2, F_2)$ respectively. Consider a circular partial finite automaton $M = (Q_\lozenge, I_\lozenge, \Sigma_\lozenge, \delta, F_\lozenge)$ where $Q_\lozenge = R_\lozenge \times S_\lozenge$, $I_\lozenge = I_1 \times I_2$ and $F_\lozenge = F_1 \times F_2$. For δ, define that for all $q_r \in R_\lozenge$, $s_r \in S_\lozenge$ and $\Sigma_\lozenge = \{x\} \cup \{\lozenge\}$, we have $\delta((q_r, s_r), (x, \lozenge)) = \delta_1(q_r, (x, \lozenge)),\ \delta_2(s_r, (x, \lozenge))$. Then $L_1(M_1) \cap L_2(M_2) = L(M)$. Therefore $L_1 \cap L_2$ is a regular circular partial language.

(ii) The proof is obvious.

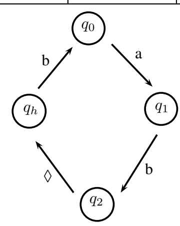
Theorem 0.8 *If the regular circular partial languages L_1 and L_2 are recognized by circular partial finite automata M_1 and M_2, then $L_1 \otimes L_2$ (length product) is also recognized by a circular partial finite automaton.*

Proof: Let L_1 and L_2 be any regular circular partial languages recognized by circular partial finite automaton $M_1 = (Q_\lozenge, I_1, \Sigma_\lozenge^1, \delta_1, F_1)$ and $M_2 = (P_\lozenge, I_2, \Sigma_\lozenge^2, \delta_2, F_2)$ respectively. Then a circular partial finite automaton $M = (R_\lozenge, I_\lozenge, \Sigma_\lozenge, \delta, F_\lozenge)$ recognizing $L_1 \otimes L_2$ is constructed such that $R_\lozenge = Q_\lozenge \times P_\lozenge$, $I_\lozenge = I_2 \times I_2$, $\Sigma_\lozenge = \Sigma_\lozenge^1 \times \Sigma_\lozenge^2$ and $F_\lozenge = F_1 \times F_2$. For δ, define it so that for all $q_r \in Q_\lozenge$, $p_r \in P_\lozenge$, $\Sigma_\lozenge^1 = \{a, b\} \cup \{\lozenge\} = \{u\} \cup \{\lozenge\}$ and $\Sigma_\lozenge^2 = \{0, 1\} \cup \{\lozenge\} = \{v\} \cup \{\lozenge\}$ we have $\delta((q_r, p_r), (w, \lozenge)) = \delta_1(q_r, (u, \lozenge)), \delta_2(p_r, (v, \lozenge))$.

Example 0.9 *Let $M_1 = (Q_\lozenge, I_1, \Sigma_\lozenge^1, \delta_1, F_1)$ with $Q_\lozenge = \{q_0, q_1, q_2, q_h\}$ and $\Sigma_\lozenge^1 = \{a, b\} \cup \{\lozenge\}$. The transition table δ_1 and transition diagram for M_1 are given as below:*

$Table\ 2$

$State$ / $Symbol$	a	b	$\Diamond$
q_0	q_1	-	-
q_1	-	q_2	-
q_2	-	-	q_h
q_h	-	q_0	-



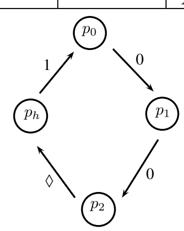
$Figure\ 3$

$$L_1(M_1) = \{(ab\Diamond b)_\circ^n | n \geq 1\}$$

Let $M_2 = (P_\Diamond, I_2, \Sigma_\Diamond^2, \delta_2, F_2)$ *with* $P_\Diamond = \{p_0, p_1, p_2, p_h\}$ *and* $\Sigma_\Diamond^2 = \{0,1\} \cup \{\Diamond\}$. *The transition table* δ_2 *and the transition diagram for* M_2 *are given below:*

$Table\ 3$

$State$ / $Symbol$	0	1	$\Diamond$
p_0	p_1	-	-
p_1	p_2	-	-
p_2	-	-	p_h
p_h	-	p_0	-



$Figure\ 4$

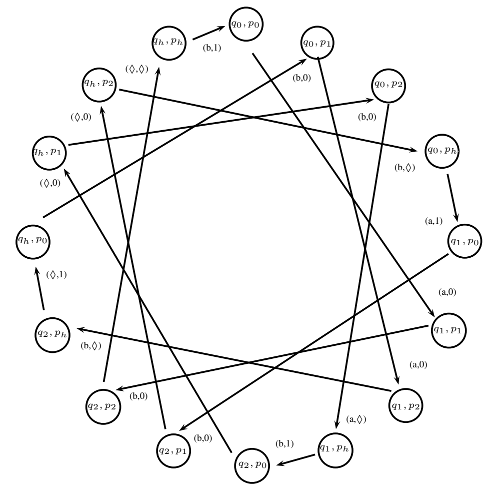
$$L_2(M_2) = \{(00\Diamond 1)_\circ^n | n \geq 1\}.$$

Now the automaton $M = (R_\Diamond, I_\Diamond, \Sigma_\Diamond, \delta, F_\Diamond)$ *recognizing* $L_1 \otimes L_2$ *is constructed such that the transition table* δ *is as follows:*

$Table\ 4$

	$(a,0)$	$(a,1)$	$(a,\Diamond)$	$(b,0)$	$(b,1)$	$(b,\Diamond)$	$(\Diamond,0)$	$(\Diamond,1)$	$(\Diamond,\Diamond)$
(q_0,p_0)	(q_1,p_1)	-	-	-	-	-	-	-	-
(q_0,p_1)	(q_1,p_2)	-	-	-	-	-	-	-	-
(q_0,p_2)	-	-	(q_1,p_h)	-	-	-	-	-	-
(q_0,p_h)	-	(q_1,p_0)	-	-	-	-	-	-	-
(q_1,p_0)	-	-	-	(q_2,p_1)	-	-	-	-	-
(q_1,p_1)	-	-	-	(q_1,p_2)	-	-	-	-	-
(q_1,p_2)	-	-	-	-	-	(q_2,p_h)	-	-	-
(q_1,p_h)	-	-	-	-	(q_2,p_0)	-	-	-	-
(q_2,p_0)	-	-	-	-	-	-	(q_h,p_1)	-	-
(q_2,p_1)	-	-	-	-	-	-	(q_h,p_2)	-	-
(q_2,p_2)	-	-	-	-	-	-	-	-	(q_h,p_h)
(q_2,p_h)	-	-	-	-	-	-	-	(q_h,p_0)	-
(q_h,p_0)	-	-	-	(q_0,p_1)	-	-	-	-	-
(q_h,p_1)	-	-	-	(q_0,p_2)	-	-	-	-	-
(q_h,p_2)	-	-	-	-	-	(q_0,p_h)	-	-	-
(q_h,p_h)	-	-	-	-	(q_0,p_0)	-	-	-	-

The transition diagram for C is as follows:

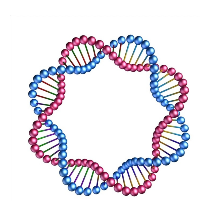


$Figure\ 5$

Artificial Intelligence and Circular Partial Words in DNA sequencing

The growth and ability of computer systems for executing tasks that usually require human intelligence is termed as Artificial intelligence (AI). In connection with clinical diagnostics, AI is defined as an accurate health data interpreter. In molecular biology, the process of establishing the order of the four nucleotide bases sequences: adenines, cytosines, thymines and guanines in a piece of DNA is termed as DNA sequencing. Nowadays with the usage of proper materials and equipment, sequencing a short piece of DNA is remarkably effortless. But sequencing a complete genome (all of an organism's DNA) still seems to be complex work. This task requires successive steps such as breaking the genomes's DNA into small pieces and then sequencing the small pieces followed by assembling the sequences into a lond single strand. Several sequencing tools are available for this process.

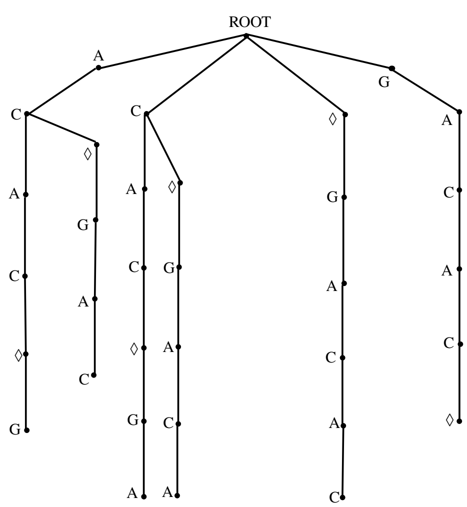
Scientists have discovered that circular DNA is commonly present in the genomes of bacteria and viruses. Of late, the study of extrachromosomal circular DNA (eccDNA) present within the nuclei of plant and animal cells has begun. In phylogenetic analysis, MtDNA are widely used.



$Figure\ 5 : Circular\ DNA$

The study and progress of circular DNA are not effectual due to the lack of constructive methods. A considerable number of algorithms for multiple sequence alignment have been proposed till date but they all handle with linear DNA and cannot deal directly circular DNA. Researchers interested in aligning circular DNA sequences must first rotate them to the "right" position using an fundamental manual process, before applying multiple sequence alignment tools.

Circularizing and rotating the circular DNA molecules prior to its alignment can remarkably enhance the efficiency of current multiple sequence alignment tools, developed for the alignment of linear DNA molecules. This pre-processing step leads to more accurate phylogenetic comparisons between species. To the best of our knowledge the CSA (Cyclic DNA Sequence Aligner tool) http://kdbio.inesc-id.pt/software/ is the only web based tool that obtains the best rotation of a set of circular DNA sequences in a very efficient way. A linear time algorithm for the construction of a generalized suffix tree for a set of sequences can be found in [Gusfield 1997]. The purpose of the algorithm is to find the best rotation among all the possible rotations of each circular genome sequence, in order to improve subsequent multiple sequence genome alignment. Unlike previous algorithms that pursued the same goal, this algorithm is handles the task effectively in linear time. This was accomplished by employing the highly efficient circular suffix tree data structure. A suffix tree for a given partial string (word) is an advanced data structure shaped like an upside down tree that stores all the suffixes of the partial string and that can be used to efficiently solve many complex partial string problems. The following figure shows a simple example of a circular suffix tree for the following sequence with hole: $ACAC\Diamond G$.



$Figure\ 6 : Circular\ Suffix\ Tree\ for\ ACAC\Diamond G$

This pre-processing step followed by the usage of CSA tool leads to more accurate phylogenetic comparisons between species.

Circular Partial Arrays

Here we define circular partial array and discuss the combinatorial properties. A circular array is the set of all conjugates of a finite array such that it is a two dimensional rectangular arrangement of letters from Σ.

Definition 0.10 *A finite circular partial array $(C_\Diamond)_\circ$ of size (m,n) is the set of all conjugates of a finite partial array $C_\Diamond$ of size (m,n) over $\Sigma_\Diamond$. Total number of distinct conjugates of the finite circular partial array is defined as the product of its number of rows and number of columns. The domain set and hole set of a finite circular partial array is unique for each one of its set of all conjugates. The set of all conjugates are segregated as column conjugates, row conjugates and column-row conjugates depending upon their rotations.*

Example 0.11 The circular partial array $(C_\Diamond)_\circ$ with distinct conjugates of a finite partial array $C_\Diamond = \begin{bmatrix} b & a & a \\ b & a & b \\ a & \Diamond & a \end{bmatrix}$ of size (3×3) over $\Sigma_\Diamond = \{a,b\} \cup \{\Diamond\}$is

$$\begin{aligned} (C_\Diamond)_\circ &= \{C_\Diamond^1, C_\Diamond^2, C_\Diamond^3, C_\Diamond^4, C_\Diamond^5, C_\Diamond^6, C_\Diamond^7, C_\Diamond^8, C_\Diamond^9\}, \\ &= \left\{ \begin{bmatrix} b & a & a \\ b & a & b \\ a & \Diamond & a \end{bmatrix}, \begin{bmatrix} a & a & b \\ a & b & b \\ \Diamond & a & a \end{bmatrix}, \begin{bmatrix} a & b & a \\ b & b & a \\ a & a & \Diamond \end{bmatrix}, \right. \\ & \begin{bmatrix} b & a & b \\ a & \Diamond & a \\ b & a & a \end{bmatrix}, \begin{bmatrix} a & b & b \\ \Diamond & a & a \\ a & a & b \end{bmatrix}, \begin{bmatrix} b & b & a \\ a & a & \Diamond \\ a & b & a \end{bmatrix}, \\ & \left. \begin{bmatrix} a & \Diamond & a \\ b & a & a \\ b & a & b \end{bmatrix}, \begin{bmatrix} \Diamond & a & a \\ a & a & b \\ a & b & b \end{bmatrix}, \begin{bmatrix} a & a & \Diamond \\ a & b & a \\ b & b & a \end{bmatrix} \right\}. \end{aligned}$$

In the above set of conjugates,
The column conjugates of $(C_\Diamond)_\circ$ are $C_\Diamond^2$ and $C_\Diamond^3$
The row conjugates of $(C_\Diamond)_\circ$ are $C_\Diamond^4$ and $C_\Diamond^7$
The column-row conjugates of $(C_\Diamond)_\circ$ are $C_\Diamond^5, C_\Diamond^6, C_\Diamond^8$and $C_\Diamond^9$.

Definition 0.12 *A weakly row period of a circular partial array $(C_\Diamond)_\circ$ of size (m,n) is a positive integer r with $P(i,j) = P(i+r,j)$ whenever (i,j) and $(i+r,j) \in D(P), 1 \le i \le m, 1 \le j \le n$. A weakly row period of the circular partial array is found using the column conjugates of the circular partial array.*

Example 0.13 *Consider a finite partial array $C_\Diamond = \begin{bmatrix} a & \Diamond & b \\ b & a & a \\ a & b & \Diamond \end{bmatrix}$ over $\Sigma_\Diamond$. Here the column conjugates of $(C_\Diamond)_\circ$ are $\begin{bmatrix} \Diamond & b & a \\ a & a & b \\ b & \Diamond & a \end{bmatrix}$ and $\begin{bmatrix} b & a & \Diamond \\ a & b & a \\ \Diamond & a & b \end{bmatrix}$. The circular partial array $(C_\Diamond)_\circ$ is weakly 2 row periodic.*

Definition 0.14 *If there exists two circular partial arrays $(A_\Diamond)_\circ$ and $(B_\Diamond)_\circ$ of equal size $(m\times n)$ with all elements of $D(A_\Diamond)_\circ$ are also in $D(B_\Diamond)_\circ$ and $(A_\Diamond)_\circ(i,j) = (B_\Diamond)_\circ(i,j)$ $\forall (i,j) \in D(A_\Diamond)_\circ$ then $(A_\Diamond)_\circ$ is said to be contained in $(B_\Diamond)_\circ$ and is represented as $(A_\Diamond)_\circ \subset (B_\Diamond)_\circ$.*

Example 0.15 *Consider two finite partial arrays* $A_\lozenge = \begin{bmatrix} a & \lozenge \\ \lozenge & b \end{bmatrix}$ *and* $B_\lozenge = \begin{bmatrix} a & \lozenge \\ b & b \end{bmatrix}$ *of size* (3×3) *over* $\Sigma_\lozenge$*. Then the circular partial arrays* $(A_\lozenge)_\circ$ *and* $(B_\lozenge)_\circ$ *with number of distinct conjugates* = 4 *are*

$$(A_\lozenge)_\circ = \{A^1_\lozenge, A^2_\lozenge, A^3_\lozenge, A^4_\lozenge\}$$

$$= \left\{ \begin{bmatrix} a & \lozenge \\ \lozenge & b \end{bmatrix}, \begin{bmatrix} \lozenge & a \\ b & \lozenge \end{bmatrix}, \begin{bmatrix} \lozenge & b \\ a & \lozenge \end{bmatrix}, \begin{bmatrix} b & \lozenge \\ \lozenge & a \end{bmatrix} \right\}.$$

$$(B_\lozenge)_\circ = \{B^1_\lozenge, B^2_\lozenge, B^3_\lozenge, B^4_\lozenge\}$$

$$= \left\{ \begin{bmatrix} a & \lozenge \\ b & b \end{bmatrix}, \begin{bmatrix} \lozenge & a \\ b & b \end{bmatrix}, \begin{bmatrix} b & b \\ a & \lozenge \end{bmatrix}, \begin{bmatrix} b & b \\ \lozenge & a \end{bmatrix} \right\}.$$

Here $(A_\lozenge)_\circ$ *is contained in* $(B_\lozenge)_\circ$ *since* $\{D(A^1_\lozenge) \subseteq D(B^1_\lozenge), D(A^2_\lozenge) \subseteq D(B^2_\lozenge), D(A^3_\lozenge) \subseteq D(B^3_\lozenge), D(A^4_\lozenge) \subseteq D(B^4_\lozenge)\}$ *and* $(A_\lozenge)_\circ(i,j) = (B_\lozenge)_\circ(i,j)\ \forall (i,j) \in D(A_\lozenge)_\circ$.

Theorem 0.16 *Let* $(C_\lozenge)_\circ$ *be a finite circular partial array and let* $(A_\lozenge)_\circ$ *and* $(B_\lozenge)_\circ$ *be two finite circular arrays. If* $(C_\lozenge)_\circ$ *has a single hole such that* $(C_\lozenge)_\circ$ *is contained in* $(A_\lozenge)_\circ(B_\lozenge)_\circ$ *as well as in* $(B_\lozenge)_\circ(A_\lozenge)_\circ$ *then* $(A_\lozenge)_\circ(B_\lozenge)_\circ = (B_\lozenge)_\circ(A_\lozenge)_\circ$.

Proof: Consider the size of the circular partial array $(A_\lozenge)_\circ$ less than or equal to $(B_\lozenge)_\circ$. Let $(B_\lozenge)_\circ = (A'_\lozenge)_\circ(B'_\lozenge)_\circ$ such that $(A'_\lozenge)_\circ, (B'_\lozenge)_\circ$ are also circular arrays and size of $(A'_\lozenge)_\circ$ is equal to that of $(A_\lozenge)_\circ$. Let $(C_\lozenge)_\circ = (D_\lozenge)_\circ(E_\lozenge)_\circ$ such that $(D_\lozenge)_\circ, (E_\lozenge)_\circ$ are also circular arrays and size of $(D_\lozenge)_\circ$ is equal to that of $(A_\lozenge)_\circ$. Then $(D_\lozenge)_\circ(E_\lozenge)_\circ \subset (A_\lozenge)_\circ(B_\lozenge)_\circ$ such that $(D_\lozenge)_\circ \subset (A_\lozenge)_\circ$ and $(E_\lozenge)_\circ \subset (B_\lozenge)_\circ$. Also since $(D_\lozenge)_\circ(E_\lozenge)_\circ \subset (B_\lozenge)_\circ(A_\lozenge)_\circ = (A'_\lozenge)_\circ(B'_\lozenge)_\circ(A_\lozenge)_\circ$, we get $(D_\lozenge)_\circ \subset (A'_\lozenge)_\circ$ and $(E_\lozenge)_\circ \subset (B'_\lozenge)_\circ(A_\lozenge)_\circ$ such that the following two cases holds.
$Case\ 1$: $(D_\lozenge)_\circ$ has null hole and $(E_\lozenge)_\circ$ has single hole. Then we get $(E_\lozenge)_\circ \subset (B_\lozenge)_\circ = (A_\lozenge)_\circ(B'_\lozenge)_\circ, (E_\lozenge)_\circ \subset (B'_\lozenge)_\circ(A_\lozenge)_\circ$ and $(D_\lozenge)_\circ = (A_\lozenge)_\circ = (A'_\lozenge)_\circ$. Then by induction process we get $(A_\lozenge)_\circ(B'_\lozenge)_\circ = (B'_\lozenge)_\circ(A_\lozenge)_\circ$ and thus $(A_\lozenge)_\circ(B_\lozenge)_\circ = (B_\lozenge)_\circ(A_\lozenge)_\circ$.
$Case\ 2$: $(D_\lozenge)_\circ$ has single hole and $(E_\lozenge)_\circ$ has null hole. Then we get $(D_\lozenge)_\circ \subset (A_\lozenge)_\circ$ and also $(D_\lozenge)_\circ \subset (A'_\lozenge)_\circ$. Also $(E_\lozenge)_\circ = (B_\lozenge)_\circ = (B'_\lozenge)_\circ(A_\lozenge)_\circ = (A'_\lozenge)_\circ(B'_\lozenge)_\circ$. Thus $(A_\lozenge)_\circ$ and $(B'_\lozenge)_\circ$ are conjugates. Consider two circular words $(U_\lozenge)_\circ$ and $(V_\lozenge)_\circ$ such that $(A_\lozenge)_\circ = (U_\lozenge)_\circ(V_\lozenge)_\circ, (A'_\lozenge)_\circ = (V_\lozenge)_\circ(U_\lozenge)_\circ$ and also $(B'_\lozenge)_\circ = ((V_\lozenge)_\circ\ (U_\lozenge)_\circ)^m\ (V_\lozenge)_\circ$ for some $m \geq 0$. Now since $(D_\lozenge)_\circ \subset (U_\lozenge)_\circ(V_\lozenge)_\circ$ and thus $(A_\lozenge)_\circ(B_\lozenge)_\circ = (B_\lozenge)_\circ(A_\lozenge)_\circ$.

Definition 0.17 *Two circular partial arrays* $(P_\lozenge)_\circ$ *and* $(Q_\lozenge)_\circ$ *are compatible (denoted as* $(P_\lozenge)_\circ \uparrow (Q_\lozenge)_\circ$*) if there exists a circular partial array* $(C_\lozenge)_\circ$ *with* $(P_\lozenge)_\circ \subset (C_\lozenge)_\circ$ *and* $(Q_\lozenge)_\circ \subset (C_\lozenge)_\circ$.

The following rules are useful for computing with circular partial arrays. Let $(P_\lozenge), (Q_\lozenge), (U_\lozenge)$ *and* $(V_\lozenge)$ *be finite partial arrays such that* $(P_\lozenge)_\circ, (Q_\lozenge)_\circ, (U_\lozenge)_\circ$ *and* $(V_\lozenge)_\circ$ *are the circular partial arrays.*

- *Multiplication: If* $(P_\lozenge)_\circ \uparrow (Q_\lozenge)_\circ$ *and* $(U_\lozenge)_\circ \uparrow (V_\lozenge)_\circ$ *then by row catenation or by column catenation,* $(P_\lozenge U_\lozenge)_\circ \uparrow (Q_\lozenge V_\lozenge)_\circ$.
- *Simplification: If* $(P_\lozenge U_\lozenge)_\circ \uparrow (Q_\lozenge V_\lozenge)_\circ$ *by row catenation or by column catenation and also if* $|(P_\lozenge)_\circ| = |(Q_\lozenge)_\circ|$ *then* $(P_\lozenge)_\circ \uparrow (Q_\lozenge)_\circ$ *and* $(U_\lozenge)_\circ \uparrow (V_\lozenge)_\circ$.
- *Weakening: If* $(P_\lozenge)_\circ \uparrow (Q_\lozenge)_\circ$ *and* $(R_\lozenge)_\circ \subset (P_\lozenge)_\circ$ *then* $(R_\lozenge)_\circ \uparrow (Q_\lozenge)_\circ$.

Example 0.18 *Consider the finite partial arrays* $P_\lozenge = \begin{bmatrix} a & \lozenge & b \\ b & b & \lozenge \\ a & \lozenge & a \end{bmatrix}$, $Q_\lozenge = \begin{bmatrix} a & a & b \\ b & b & \lozenge \\ a & \lozenge & a \end{bmatrix}$, $R_\lozenge = \begin{bmatrix} a & \lozenge & b \\ \lozenge & b & \lozenge \\ a & \lozenge & a \end{bmatrix}$ *of size* (3×3) *over the alphabet* $\Sigma_\lozenge$*. Let* $U_\lozenge = \begin{bmatrix} a & a & \lozenge \\ b & a & \lozenge \end{bmatrix}$, $V_\lozenge = \begin{bmatrix} a & a & b \\ b & a & \lozenge \end{bmatrix}$ *be two finite partial arrays of size* (2×3)*. Then the circular partial arrays* $(P_\lozenge)_\circ$,$(Q_\lozenge)_\circ$ *and* $(R_\lozenge)_\circ$ *with number of distinct conjugates* = 9 *are*

$$(P_\lozenge)_\circ = \{P^1_\lozenge, P^2_\lozenge, P^3_\lozenge, P^4_\lozenge, P^5_\lozenge, P^6_\lozenge, P^7_\lozenge, P^8_\lozenge, P^9_\lozenge\}$$

$$= \left\{ \begin{bmatrix} a & \lozenge & b \\ b & b & \lozenge \\ a & \lozenge & a \end{bmatrix}, \begin{bmatrix} \lozenge & b & a \\ b & \lozenge & b \\ \lozenge & a & a \end{bmatrix}, \begin{bmatrix} b & a & \lozenge \\ \lozenge & b & b \\ a & a & \lozenge \end{bmatrix}, \right.$$

$$\begin{bmatrix} b & b & \lozenge \\ a & \lozenge & a \\ a & \lozenge & b \end{bmatrix}, \begin{bmatrix} b & \lozenge & b \\ \lozenge & a & a \\ \lozenge & b & a \end{bmatrix}, \begin{bmatrix} \lozenge & b & b \\ a & a & \lozenge \\ b & a & \lozenge \end{bmatrix},$$

$$\left. \begin{bmatrix} a & \lozenge & a \\ a & \lozenge & b \\ b & b & \lozenge \end{bmatrix}, \begin{bmatrix} \lozenge & a & a \\ \lozenge & b & a \\ b & \lozenge & b \end{bmatrix}, \begin{bmatrix} a & a & \lozenge \\ b & a & \lozenge \\ \lozenge & b & b \end{bmatrix} \right\}.$$

$$(Q_\lozenge)_\circ = \{Q^1_\lozenge, Q^2_\lozenge, Q^3_\lozenge, Q^4_\lozenge, Q^5_\lozenge, Q^6_\lozenge, Q^7_\lozenge, Q^8_\lozenge, Q^9_\lozenge\}$$

$$= \left\{ \begin{bmatrix} a & a & b \\ b & b & \lozenge \\ a & \lozenge & a \end{bmatrix}, \begin{bmatrix} a & b & a \\ b & \lozenge & b \\ \lozenge & a & a \end{bmatrix}, \begin{bmatrix} b & a & a \\ \lozenge & b & b \\ a & a & \lozenge \end{bmatrix}, \right.$$

$$\begin{bmatrix} b & b & \lozenge \\ a & \lozenge & a \\ a & a & b \end{bmatrix}, \begin{bmatrix} b & \lozenge & b \\ \lozenge & a & a \\ a & b & a \end{bmatrix}, \begin{bmatrix} \lozenge & b & b \\ a & a & \lozenge \\ b & a & a \end{bmatrix},$$

$$\left. \begin{bmatrix} a & \lozenge & a \\ a & a & b \\ b & b & \lozenge \end{bmatrix}, \begin{bmatrix} \lozenge & a & a \\ a & b & a \\ b & \lozenge & b \end{bmatrix}, \begin{bmatrix} a & a & \lozenge \\ b & a & a \\ \lozenge & b & b \end{bmatrix} \right\}.$$

and

$$(R_\lozenge)_\circ = \{R^1_\lozenge, R^2_\lozenge, R^3_\lozenge, R^4_\lozenge, R^5_\lozenge, R^6_\lozenge, R^7_\lozenge, R^8_\lozenge, R^9_\lozenge\}$$

$$= \left\{ \begin{bmatrix} a & \lozenge & b \\ \lozenge & b & \lozenge \\ a & \lozenge & a \end{bmatrix}, \begin{bmatrix} & b & a \\ b & \lozenge & \lozenge \\ \lozenge & a & a \end{bmatrix}, \begin{bmatrix} b & a & \lozenge \\ \lozenge & \lozenge & b \\ a & a & \lozenge \end{bmatrix}, \right.$$

$$\begin{bmatrix} \Diamond & b & \Diamond \\ a & \Diamond & a \\ a & \Diamond & b \end{bmatrix}, \begin{bmatrix} b & \Diamond & \Diamond \\ \Diamond & a & a \\ \Diamond & b & a \end{bmatrix}, \begin{bmatrix} \Diamond & \Diamond & b \\ a & a & \Diamond \\ b & a & \Diamond \end{bmatrix},$$

$$\begin{bmatrix} a & \Diamond & a \\ a & \Diamond & b \\ \Diamond & b & \Diamond \end{bmatrix}, \begin{bmatrix} \Diamond & a & a \\ \Diamond & b & a \\ b & \Diamond & \Diamond \end{bmatrix}, \begin{bmatrix} a & a & \Diamond \\ b & a & \Diamond \\ \Diamond & \Diamond & b \end{bmatrix} \Bigg\}.$$

The circular partial arrays $(U_\Diamond)_\circ$ and $(V_\Diamond)_\circ$ with number of distinct conjugates = 6 are

$$(U_\Diamond)_\circ = \{U^1_\Diamond, U^2_\Diamond, U^3_\Diamond, U^4_\Diamond, U^5_\Diamond, U^6_\Diamond\}$$

$$= \Bigg\{ \begin{bmatrix} a & a & \Diamond \\ b & a & \Diamond \end{bmatrix}, \begin{bmatrix} a & \Diamond & a \\ a & \Diamond & b \end{bmatrix}, \begin{bmatrix} \Diamond & a & a \\ \Diamond & b & a \end{bmatrix},$$

$$\begin{bmatrix} b & a & \Diamond \\ a & a & \Diamond \end{bmatrix}, \begin{bmatrix} a & \Diamond & b \\ a & \Diamond & a \end{bmatrix}, \begin{bmatrix} \Diamond & b & a \\ \Diamond & a & a \end{bmatrix} \Bigg\}$$

and

$$(V_\Diamond)_\circ = \{V^1_\Diamond, V^2_\Diamond, V^3_\Diamond, V^4_\Diamond, V^5_\Diamond, V^6_\Diamond\}$$

$$= \Bigg\{ \begin{bmatrix} a & a & b \\ b & a & \Diamond \end{bmatrix}, \begin{bmatrix} a & b & a \\ a & \Diamond & b \end{bmatrix}, \begin{bmatrix} b & a & a \\ \Diamond & b & a \end{bmatrix},$$

$$\begin{bmatrix} b & a & \Diamond \\ a & a & b \end{bmatrix}, \begin{bmatrix} a & \Diamond & b \\ a & b & a \end{bmatrix}, \begin{bmatrix} \Diamond & b & a \\ b & a & a \end{bmatrix}. \Bigg\}$$

Here we have $(P_\Diamond)_\circ \uparrow (Q_\Diamond)_\circ$ and $(U_\Diamond)_\circ \uparrow (V_\Diamond)_\circ$ since the conjugates

$$\{P^1_\Diamond \uparrow Q^1_\Diamond, P^2_\Diamond \uparrow Q^2_\Diamond, ...P^9_\Diamond \uparrow Q^9_\Diamond\}$$

and

$$\{U^1_\Diamond \uparrow V^1_\Diamond, U^2_\Diamond \uparrow V^2_\Diamond, ...U^6_\Diamond \uparrow V^6_\Diamond\}.$$

Then by row catenation we get $(P_\Diamond U_\Diamond)_\circ \uparrow (Q_\Diamond V_\Diamond)_\circ$ such that

$$\left\{(P_\Diamond U_\Diamond)^1 \uparrow (Q_\Diamond V_\Diamond)^1, ..., (P_\Diamond U_\Diamond)^{15} \uparrow Q_\Diamond V_\Diamond)^{15}\right\}.$$

By Definition 0.10 we get $(R_\Diamond)_\circ \subset (P_\Diamond)_\circ$ since

$$\left\{D(R^1_\Diamond) \subseteq D(P^1_\Diamond), ..., D(R^9_\Diamond) \subseteq D(P^9_\Diamond)\right\}$$

and $(R_\Diamond)_\circ(i,j) = (P_\Diamond)_\circ(i,j)\ \forall (i,j) \in D(R_\Diamond)_\circ$. Hence we get $(R_\Diamond)_\circ \uparrow (Q_\Diamond)_\circ$.

Theorem 0.19 *Levi's Theorem [**?**]: Consider the partial words u, v, w and z. If $uv \uparrow wz$ and $|u| \le |w|$ then a factorisation $w = yz$ exists with $u \uparrow z$ and $v \uparrow yz$.*

Theorem 0.20 *Consider $P_\Diamond$ and $Q_\Diamond$ to be two non empty finite partial arrays forming the circular partial arrays $(P_\Diamond)_\circ$ and $(P_\Diamond)_\circ$ with $(P_\Diamond Q_\Diamond)_\circ$ having a single hole. $(P_\Diamond Q_\Diamond)_\circ \uparrow (Q_\Diamond P_\Diamond)_\circ$ iff $(P_\Diamond)_\circ \subset (R^m_\Diamond)_\circ, (Q_\Diamond)_\circ \subset (R^n_\Diamond)_\circ$ for some array $R_\Diamond$ and integers m, n.*

Proof: Consider the non-empty finite partial arrays $P_\Diamond$ and $Q_\Diamond$.
Assumption 1: If $|(P_\Diamond)_\circ| = |(Q_\Diamond)_\circ|$. Then by $Simplification\ Rule$, we get $(P_\Diamond)_\circ \uparrow (Q_\Diamond)_\circ$. Since $(P_\Diamond Q_\Diamond)_\circ$ has a single hole, either $(P_\Diamond)_\circ$ or $(Q_\Diamond)_\circ$ has a single hole. This shows that either $(P_\Diamond)_\circ \subset (Q_\Diamond)_\circ$. or $(Q_\Diamond)_\circ \subset (P_\Diamond)_\circ$.
Assumption 2: If $|(P_\Diamond)_\circ| < |(Q_\Diamond)_\circ|$. By Levi's theorem, there exists a factorisation $(Q_\Diamond)_\circ = (A_\Diamond B_\Diamond)_\circ$ such that $|(A_\Diamond)_\circ| = |(P_\Diamond)_\circ|, (P_\Diamond)_\circ \uparrow (A_\Diamond)_\circ$ and $(A_\Diamond B_\Diamond)_\circ \uparrow (B_\Diamond P_\Diamond)_\circ$. Now three cases arise.
$Case$ 1: If $(P_\Diamond)_\circ$ is the circular partial array with single hole. Then $(P_\Diamond)_\circ \uparrow (A_\Diamond)_\circ \Rightarrow (P_\Diamond)_\circ \subset (A_\Diamond)_\circ$ and hence $(P_\Diamond B_\Diamond)_\circ \subset (A_\Diamond B_\Diamond)_\circ$. By $Weakening\ Rule$, we get $(P_\Diamond B_\Diamond)_\circ \uparrow (B_\Diamond P_\Diamond)_\circ$. Also since $(P_\Diamond)_\circ \subset (A_\Diamond)_\circ$ we get $(P_\Diamond B_\Diamond)_\circ \subset (B_\Diamond A_\Diamond)_\circ$. By Theorem 0.16 $(A_\Diamond B_\Diamond)_\circ \subset (B_\Diamond A_\Diamond)_\circ$ and there exists array $(R_\Diamond)_\circ$ such that $(B_\Diamond)_\circ = (R^m_\Diamond)_\circ$ and $(A_\Diamond)_\circ = (R^n_\Diamond)_\circ$ and also $(Q_\Diamond)_\circ = (R^{m+n}_\Diamond)_\circ$ and $(P_\Diamond)_\circ \subset (R^m_\Diamond)_\circ$.
$Case$ 2: If $(A_\Diamond)_\circ$ is the circular partial array with single hole. Then this case is symmetric to case 1.
Case 3: If $(B_\Diamond)_\circ$ is the circular partial array with single hole. Then since $(P_\Diamond)_\circ \uparrow (A_\Diamond)_\circ$ and $(A_\Diamond B_\Diamond)_\circ \uparrow (B_\Diamond P_\Diamond)_\circ$ we get $(P_\Diamond)_\circ = (A_\Diamond)_\circ$ and $(P_\Diamond B_\Diamond)_\circ \uparrow (B_\Diamond A_\Diamond)_\circ$. By induction process, $(P_\Diamond)_\circ \subset (R^m_\Diamond)_\circ$ and $(B_\Diamond)_\circ \subset (R^n_\Diamond)_\circ$. Then by $Multiplication\ Rule$ $(Q_\Diamond)_\circ = (A_\Diamond B_\Diamond)_\circ \subset (R^{m+n}_\Diamond)_\circ$.

Conversely, consider $(P_\Diamond)_\circ \subset (R^m_\Diamond)_\circ, (Q_\Diamond)_\circ \subset (R^n_\Diamond)_\circ$ for some m, n. By $Multiplication\ Rule$ we get, $(P^n_\Diamond)_\circ \subset (R^{mn}_\Diamond)_\circ$ and $(Q^m_\Diamond)_\circ \subset (R^{mn}_\Diamond)_\circ$ which shows that $(P_\Diamond Q_\Diamond)_\circ \uparrow (Q_\Diamond P_\Diamond)_\circ$.

Definition 0.21 *Two circular partial arrays $(P_\Diamond)_\circ$ and $(Q_\Diamond)_\circ$ of size $(m \times n)$are called conjugates if partial arrays (or total arrays) $(U_\Diamond)$ and $(V_\Diamond)$ exists with $(P_\Diamond)_\circ \subset (U_\Diamond V_\Diamond)_\circ$ and $(Q_\Diamond)_\circ \subset (V_\Diamond U_\Diamond)_\circ$. We use row catenation or column catenation to show the conjugate relation.*

Definition 0.22 *Consider $P_\Diamond$ and $Q_\Diamond$ to be finite partial arrays with $P_\Diamond \uparrow Q_\Diamond$. The least upper bound of circular partial arrays $(P_\Diamond)_\circ$ and $(Q_\Diamond)_\circ$ is the circular partial array $(P_\Diamond)_\circ \vee (Q_\Diamond)_\circ$, where $(P_\Diamond)_\circ$ is contained in $(P_\Diamond)_\circ \vee (Q_\Diamond)_\circ$ and $(Q_\Diamond)_\circ$ is contained in $(P_\Diamond)_\circ \vee (Q_\Diamond)_\circ$ and also $D((P_\Diamond)_\circ \vee (Q_\Diamond)_\circ) = D((P_\Diamond)_\circ \cup D((Q_\Diamond)_\circ$.*

Theorem 0.23 *If two circular partial arrays $(P_\Diamond)_\circ$ and $(Q_\Diamond)_\circ$ of size $(m \times n)$ are conjugates then there exists a non empty finite partial array $R_\Diamond$ such that $(P_\Diamond R_\Diamond)_\circ \uparrow (R_\Diamond Q_\Diamond)_\circ$ where $P_\Diamond$ and $Q_\Diamond$ represent finite partial arrays.*

Proof: Let $(P_\Diamond)_\circ$ and $(Q_\Diamond)_\circ$ be two finite circular partial arrays that are conjugate to each other. Consider any two non-empty partial arrays $U_\Diamond$ and $V_\Diamond$ such that $(P_\Diamond)_\circ$ is contained in $(U_\Diamond V_\Diamond)_\circ$ and $(Q_\Diamond)_\circ$ is contained in $(V_\Diamond U_\Diamond)_\circ$ by either row catenation or column catenation. Then $(P_\Diamond U_\Diamond)_\circ$ is contained in $(U_\Diamond V_\Diamond U_\Diamond)_\circ$ and $(U_\Diamond Q_\Diamond)_\circ$ is contained in $(U_\Diamond V_\Diamond U_\Diamond)_\circ$. Thus for $R_\Diamond = U_\Diamond$, we have $(P_\Diamond R_\Diamond)_\circ \uparrow (R_\Diamond Q_\Diamond)_\circ$.

Theorem 0.24 *Consider two non empty finite circular partial arrays $(P_\Diamond)_\circ$ and $(Q_\Diamond)_\circ$ formed from the finite partial* *arrays $P_\Diamond$ and $Q_\Diamond$ over $\Sigma_\Diamond$. If there exists a finite partial array $R_\Diamond$ with $(P_\Diamond R_\Diamond)_\circ \uparrow (R_\Diamond Q_\Diamond)_\circ$ and $(P_\Diamond R_\Diamond)_\circ \vee (R_\Diamond Q_\Diamond)_\circ$ is $|(P_\Diamond)_\circ|$- weakly row periodic, then there exists partial arrays $U_\Diamond$ and $V_\Diamond$ such that $(P_\Diamond)_\circ \subset (U_\Diamond V_\Diamond)_\circ, (Q_\Diamond)_\circ \subset (V_\Diamond U_\Diamond)_\circ$ and $(R_\Diamond)_\circ \subset (U_\Diamond(V_\Diamond U_\Diamond)^m)_\circ$ for some $m \geq 0$.*

Proof: Let $(P_\Diamond)_\circ$ and $(Q_\Diamond)_\circ$ be finite circular partial arrays. Let $R_\Diamond, U_\Diamond$ and $V_\Diamond$ be any non empty finite partial arrays such that by Theorem 0.23, we get $P_\Diamond R_\Diamond \uparrow R_\Diamond Q_\Diamond$ and also $(P_\Diamond)_\circ \subset (U_\Diamond V_\Diamond)_\circ$ and $(Q_\Diamond)_\circ \subset (V_\Diamond U_\Diamond)_\circ$ by either row or column catenation. Now we assume that $(P_\Diamond R_\Diamond)_\circ \vee (R_\Diamond Q_\Diamond)_\circ$ with the arrays $P_\Diamond R_\Diamond$ and $R_\Diamond Q_\Diamond$ formed by either row or column catenation are $|(P_\Diamond)_\circ|$- weakly row periodic. Consider t with

$$t|P_\Diamond| > |R_\Diamond| \geq (t-1)|P_\Diamond|$$

Consider $(P_\Diamond)_\circ = (U^1_\Diamond V^1_\Diamond)_\circ$ and $(Q_\Diamond)_\circ = (V^2_\Diamond U^2_\Diamond)_\circ$ with

$$\begin{aligned} |(U^1_\Diamond)_\circ| &= |(U^2_\Diamond)_\circ| = |(R_\Diamond)_\circ| - (t-1)|(P_\Diamond)_\circ| \\ |(V^1_\Diamond)_\circ| &= |(V^2_\Diamond)_\circ| \end{aligned}$$

Consider

$(R_\Diamond)_\circ = ((U^1_\Diamond)'(V^1_\Diamond)'(U^2_\Diamond)'....(U^{m-1}_\Diamond)'(V^{m-1}_\Diamond)'(U^m_\Diamond)')_\circ$ such that

$$\begin{aligned} |((U^1_\Diamond)')_\circ| &= |((U^2_\Diamond)')_\circ| = |((U^{m-1}_\Diamond)')_\circ| = |((U^m_\Diamond)')_\circ| \\ &= |(U^1_\Diamond)_\circ| = |(U^2_\Diamond)_\circ| \\ |((V^1_\Diamond)')_\circ| &= |((V^2_\Diamond)')_\circ| = |((V^{m-1}_\Diamond)')_\circ| \\ &= |(V^1_\Diamond)_\circ| = |(V^2_\Diamond)_\circ|. \end{aligned}$$

Since $(P_\Diamond R_\Diamond)_\circ \uparrow (R_\Diamond Q_\Diamond)_\circ$, we can derive that $U^1_\Diamond \subset U_\Diamond$ and $U^2_\Diamond \subset U_\Diamond$. Also $V^1_\Diamond \subset V_\Diamond$ and $V^2_\Diamond \subset V_\Diamond$. Thus we get

$$\begin{aligned} (P_\Diamond)_\circ &= (U^1_\Diamond V^1_\Diamond)_\circ \subset (U_\Diamond V_\Diamond)_\circ \\ (Q_\Diamond)_\circ &= (V^2_\Diamond U^2_\Diamond)_\circ \subset (V_\Diamond U_\Diamond)_\circ \end{aligned}$$

Further

$(R_\Diamond)_\circ = ((U^1_\Diamond)'(V^1_\Diamond)'(U^2_\Diamond)'...(U^{m-1}_\Diamond)'(V^{m-1}_\Diamond)'(U^m_\Diamond)')_\circ$ is contained in $(U_\Diamond(V_\Diamond U_\Diamond)^m)_\circ$ for some $m \geq 0$ and hence the result.

Definition 0.25 *A circular partial array $(P_\Diamond)_\circ$ is said to be primitive if there exists no circular partial array $(Q_\Diamond)_\circ$ such that $(P_\Diamond)_\circ \subset (Q^m_\Diamond)_\circ, m \geq 2$. $(P_\Diamond)_\circ$ is primitive whenever the primitive circular partial array $(Q_\Diamond)_\circ$ is contained in $(P_\Diamond)_\circ$. $(P_\Diamond)_\circ$ has $(m \times n)$ number of distinct conjugates.*

Example 0.26 *The circular partial array $(P_\Diamond)_\circ$ over $\Sigma_\Diamond$ of the non-empty finite partial array $P_\Diamond = \begin{bmatrix} a & \Diamond & b & a \\ b & a & a & a \end{bmatrix}$ is primitive with 8 distinct conjugates.*

The circular partial array $(Q_\Diamond)_\circ$ of the non-empty finite partial array $Q_\Diamond = \begin{bmatrix} a & a & \Diamond & a \\ b & a & b & a \end{bmatrix}$ is not primitive with repetitive conjugates.

Theorem 0.27 *Consider two finite circular partial arrays $(P_\Diamond)_\circ$ and $(Q_\Diamond)_\circ$ formed from the finite partial arrays $P_\Diamond$ and $Q_\Diamond$. If $(P_\Diamond Q_\Diamond)_\circ$ is primitive then $(Q_\Diamond P_\Diamond)_\circ$ is also primitive.*

Proof: Let us prove by contradiction. Assume that $(P_\Diamond Q_\Diamond)_\circ$ is primitive but $(Q_\Diamond P_\Diamond)_\circ$ is not primitive. Then by Definition 0.25, let $(B)_\circ$ be any finite circular array such that $(Q_\Diamond P_\Diamond)_\circ \subset (B^n_\Diamond)_\circ$ for some $n \geq 2$. Then there exists circular arrays $(B_1)_\circ, (B_2)_\circ$ formed from finite arrays (B_1) and (B_2), such that $(B)_\circ = (B_1 B_2)_\circ$, $(Q_\Diamond)_\circ \subset ((B_1 B_2)^r B_1)_\circ$ and $(P_\Diamond)_\circ \subset (B_2(B_1 B_2)^s)_\circ$ with $n - 1 = r + s$. Then $(P_\Diamond Q_\Diamond)_\circ \subset (B_2 B_1)^n_\circ$ which follows that $P_\Diamond Q_\Diamond$ is not primitive. Therefore if $(P_\Diamond Q_\Diamond)_\circ$ is primitive then $(Q_\Diamond P_\Diamond)_\circ$ is also primitive.

Theorem 0.28 *Consider two finite circular partial arrays $(P_\Diamond)_\circ$ and $(Q_\Diamond)_\circ$. If there exists a primitive circular array $(A)_\circ$ where $(P_\Diamond Q_\Diamond)_\circ \subset (A^n)_\circ$ for some $n \geq 1$ then a primitive circular array $(B)_\circ$ exists such that $(Q_\Diamond P_\Diamond)_\circ \subset (B^n)_\circ$ where $P_\Diamond$ and $Q_\Diamond$ are finite partial arrays.*

Proof: Assume a primitive circular array $(A)_\circ$ with $(P_\Diamond Q_\Diamond)_\circ \subset (A^n)_\circ$ for some $n \geq 1$.
$Case$ 1: If $n = 1$. Then $(P_\Diamond Q_\Diamond)_\circ \subset (A)_\circ$. Assume that $(A)_\circ = (A'B')_\circ$ with $|(A)_\circ| = |(A)'_\circ|$ and $|(B)_\circ| = |(B)'_\circ|$. Then the result follows whenever $(B)_\circ = (B'A')_\circ$ since $(A'B')_\circ$ is primitive.
$Case$ 2: If $n > 1$. Assume the partial arrays A_1, A_2 such that $A = A_1 A_2$, $(P_\Diamond)_\circ \subset ((A_1 A_2)^r A_1)_\circ$ and $(Q_\Diamond)_\circ \subset (A_2(A_1 A_2)^s)_\circ$ with $n - 1 = r + s$. Since $(A)_\circ$ is primitive, $A_1 A_2$ as well as $A_2 A_1$ are also primitive. Therefore the result follows since $(P_\Diamond Q_\Diamond)_\circ \subset (A_2 A_1)^n_\circ$.

Conclusion

Here we extended circular language to circular partial language. Further we studied about their closure properties. Also we extended circular partial words to arrays and discussed certain combinatorics such as periodicity, compatibility, conjugacy, commutativity and primitivity. In future properties of pcodes, correlations, borderness property etc can be studied.

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