Quadratic reformulations for the optimization of pseudo-boolean functions

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We investigate various solution approaches for the unconstrained minimization of a pseudo-boolean function. More precisely, we assume that the original function is expressed as a real-valued polynomial in 0-1 variables, of degree three or more, and we consider a generic family of two-step approaches for its minimization. First, a *quadratic reformulation step* aims at transforming the minimization problem into an equivalent constrained or unconstrained quadratic 0-1 minimization problem (where "equivalent" means here that a minimizer of the original function can be easily deduced from a minimizer of the reformulation). Second, an *optimization step* handles the obtained equivalent quadratic problem.

We provide a unified presentation of several quadratic reformulation schemes proposed in the literature, e.g., (Anthony et al. 2017; Buchheim and Rinaldi 2007; Rodríguez-Heck 2018; Rosenberg 1975), and we review several methods that can be applied in the optimization step, including a standard linearization procedure (Fortet 1959) and more elaborate convex quadratic reformulations, as in (Billionnet and Elloumi 2007; Billionnet, Elloumi, and Lambert 2012, 2016; Elloumi, Lambert, and Lazare 2021). We discuss the impact of the reformulation scheme on the efficiency of the optimization step and we illustrate our discussion with some computational results on different classes of instances.

References

Anthony, M.; Boros, E.; Crama, Y.; and Gruber, A. 2017. Quadratic reformulations of nonlinear binary optimization problems. *Mathematical Programming* 162(1-2): 115–144.

Billionnet, A.; and Elloumi, S. 2007. Using a mixed integer quadratic programming solver for the unconstrained quadratic 0–1 problem. *Mathematical Programming* 109(1): 55–68.

Billionnet, A.; Elloumi, S.; and Lambert, A. 2012. Extending the QCR method to general mixed-integer programs. *Mathematical programming* 131(1–2): 381–401.

Billionnet, A.; Elloumi, S.; and Lambert, A. 2016. Exact quadratic convex reformulations of mixed-integer quadratically constrained problems. *Mathematical Programming* 158(1–2): 235–266.

Buchheim, C.; and Rinaldi, G. 2007. Efficient reduction of polynomial zero-one optimization to the quadratic case. *SIAM Journal on Optimization* 18(4): 1398–1413.

Elloumi, S.; Lambert, A.; and Lazare, A. 2021. Solving unconstrained 0–1 polynomial programs through quadratic convex reformulation. *Journal of Global Optimization* 80(2): 231–248.

Fortet, R. 1959. L'algèbre de Boole et ses applications en recherche opérationnelle. *Cahiers du Centre d'Etudes de Recherche Opérationnelle* 4: 5–36.

Rodríguez-Heck, E. 2018. *Linear and quadratic reformulations of nonlinear optimization problems in binary variables*. Ph.D. thesis. University of Liège.

Rosenberg, I. G. 1975. Reduction of bivalent maximization to the quadratic case. *Cahiers du Centre d'Études de Recherche Opérationnelle* 17: 71–74.