Boole's probability bounding problem, linear programming aggregations, and nonnegative quadratic pseudo-Boolean functions

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Abstract

George Boole proposed the union bounding problem, which is a class of probabilistic satisfiability/optimization problems. In its most frequently studied case, we are given the probabilities of n events and also the probabilities of their pairwise intersections, and want to know how small/large can the probability of their union be. Hailperin (1965) provided a linear programming model for this problem which involves $2^n - 1$ variables. The model generates tight bounds (both the minimum and the maximum are realizable by events) but it may be hard to compute. Large number of papers were published in the last 100 years that provide upper and/or lower bounds for this problem. Many are based on Hailperin's model and considers a relaxation that either has polynomial dimension in n or at least is computable in time polynomial in n.

We study methods that aggregate subsets of variables in Hailperin's model in various ways. These aggregation methods can be viewed as monotone linear mappings $\mathcal{L}$ to a lower N dimensional space such that $\mathcal{L}(\mathbb{R}_+^{2^n-1}) \subseteq \mathbb{R}_+^N$. While several publications in the literature considered this type of relaxations of Hailperin's model, only few noted that we may have strict containment in the above relation. We provide a complete characterization of the cone $\mathcal{L}(\mathbb{R}_+^{2^n-1})$ for three different aggregation models from the literature – the so called binomial moment problem (see e.g., Prékopa (1988)), the relaxation by Prékopa and Gao (2005), and the model by Yang, Alajaji, and Takahara (2018). These characterizations lead to strictly improved bounds in the last two of these cases. We also obtain an $O(n^2)$ dimensional polyhedral formulation that is equivalent with Hailperin's model, simultaneously for minimization and maximization, unlike the ones obtained by linear programming duality. An interesting side result is that there exists a new connection between Hailperin's model and the Boolean Quadric polytope (see e.g., Padberg (1985)).